0. Ch3.1 A trivial proof and a vacuous proof (Reading assignment)
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## Ch 3.2: Direct proofs

A direct proof is a way of showing that a given statement is true or false by using existing lemmas and theorems without making any further assumptions. To prove statements of the form "if $P$, then $Q$",

Assume that **the statement $P$ is true** and directly derive the conclusion that **the statement $Q$ is true**.

We can use the following properties of integers without justification.

- The sum (difference, product) of every two integers is an integer.
- The product of two negative integer is positive.
- Every integer is of the form 2m or 2m+1, where $m \in \mathbb{Z}$.

$\vdots$

**Definition**: An integer x is called even (respectively odd) if there is an integer k for which x = 2k (respectively 2k+1).

**Example**. If $n$ is an even integer, then $7n + 4$ is also an even integer.

Write a *hypothesis* and a *conclusion* first and fill out *the body of the proof* which is a bridge of logical deductions from the hypothesis to the conclusion.

*Proof.*

**Exercises**

1. Let $n \in \mathbb{Z}$. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

2. Let $S = \{0, 1, 2\}$ and let $n \in S$. Prove that if $(n+1)^2(n+2)^2/4$ is even, then $(n+2)^2(n+3)^2/4$ is even.

## Proofs Involving Inequalities

(A1) For all real numbers $a, b, c$, if $a \leq b$ and $b \leq c$ then $a \leq c$.

(A2) For all real numbers $a, b, c$, if $a \leq b$ then $a + c \leq b + c$.

(A3) For all real numbers $a, b, c$, if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.

Prove the statements below using A1-A3, together with any basic facts about *equality* $=$.

(1) For all real numbers $a, b$, if $0 \leq a$ and $a \leq b$ then $a^2 \leq b^2$.

(2) For all real numbers $a$, if $a \leq 0$ then $0 \leq -a$.

(3) For all real numbers $a, b$, if $b \leq a$ and $a \leq 0$, then $a^2 \leq b^2$.

(4) For all real numbers $b$, $0 \leq b^2$.

(5) For all real numbers $a, b$, $ab \leq \frac{1}{2}(a^2 + b^2)$. *Hint: Consider* $(a - b)^2$.

(6) For all real numbers $a, b, \delta$, if $\delta \neq 0$ then $ab \leq \frac{1}{2}(\delta^2 a^2 + \delta^{-2} b^2)$.

(7) For all real numbers $a, b$, $ab = \frac{1}{2}(a^2 + b^2)$ if and only if $a = b$.

## Working Backwards

**Theorem** ( Inequality between arithmetic and geometric mean. )

$$\text{If } a, b \in \mathbb{R} \text{ are such that } a \geq 0 \text{ and } b \geq 0, \text{ then } \frac{a+b}{2} \geq \sqrt{ab}$$

**Scratch work:**

1. Start with the inequality you are asked to prove.
2. Simplify it as much as possible until you arrive at a statement that is obviously true.

**Formal Proof:**

3. In order to write formal proof, now start from the obviously true statement.
4. Use your previous work to guide you on how to arrive at the desired inequality.

*Proof*:

## Common mistakes.

1. **What is wrong with this proof?**
   (1) Assume $a = b$.
   (2) Multiplying both sides by $b$, $ab = b^2$.
   (3) Subtracting $a^2$ from both sides, $ab - a^2 = b^2 - a^2$.
   (4) Factoring $a(b - a) = (b + a)(b - a)$.
   (5) Dividing by $b - a$, $a = b + a$.
   (6) Using (1), $a = 2a$.
   (7) Dividing by $a$, $1 = 2$.

2. **Circular argument** Prove if $n^3$ is even then $n$ is even.

   *Proof*:

   Assume $n^3$ is even.
   Then $\exists k \in \mathbb{Z}$ such that $n^3 = 8k^3$.
   It follows that $n = \left(8k^3\right)^{1/3} = 2k$.
   Therefore $n$ is even.

   All statements in the proof are true but is the proof correct?

## Ch 3.3: Proof by contrapositive

It is a direct proof but we start with the contrapositive because

$$P \implies Q \text{ is equivalent to } \sim(Q) \implies \sim(P).$$

**Why** do we prove the contrapositive of the implication instead of the original implication?

**Example**. Prove: If $n^3$ is even then $n$ is even.

**Definition**: Two integers are said to have the same parity if they are both odd or both even.

**Theorem**: Let $x, y \in \mathbb{Z}$. Then $x$ and $y$ are of the same parity if and only if $x + y$ is even.

*Proof.*

**Exercises**

1. Prove that if $5x - 11$ is an even integer, then $x$ is an odd integer.

2. Prove that if $7x + 4$ is even, then $5x - 11$ is odd.
   (This problem is a continuation of the previous question. Of course we can prove this statement directly by using the definition of an even and odd integers)